Homework III
Due Date: 20/04/2023

Exercise 1. Consider the following three Cauchy problems.
(i) (1 point) Solve

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = tx, & \text{for } (t,x) \in [0,\infty) \times \mathbb{R}, \\ (u, \partial_t u)_{|t=0} = (0,0), & \text{for } x \in \mathbb{R}. \end{cases}$$

(ii) (1 point) Solve

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = e^{ax}, & \text{for } (t,x) \in [0,\infty) \times \mathbb{R}, \\ (u, \partial_t u)_{|t=0} = (0,0), & \text{for } x \in \mathbb{R}. \end{cases}$$

(iii) (1 point) Solve

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = \cos x, & \text{for } (t,x) \in [0,\infty) \times \mathbb{R}, \\ (u, \partial_t u)_{|t=0} = (\sin x, 1+x), & \text{for } x \in \mathbb{R}. \end{cases}$$

Exercise 2. We consider a C^2 solution u of the 1D wave equation

$$\partial_t^2 u - \partial_x^2 u = 0,$$

in the cylinder $\mathcal{C} = \left\{(t,x) \in \mathbb{R}^2 : (t,x) \in [0,\infty) \times [a,b]\right\}$ with $-\infty < a < b < +\infty$. Assume that u satisfies the boundary condition

$$u(t,a) = 0 \quad \text{and} \quad (\partial_t u + \partial_x u)(t,b) = 0, \quad \text{for all } t \geq 0.$$

(i) (1 point) Define the energy of u at time t by

$$E(t) = \frac{1}{2} \int_a^b \left[(\partial_t u)^2 + (\partial_x u)^2\right](t,x) \mathrm{d}x.$$

Show that

$$E(T) - E(0) = -\int_0^T (\partial_t u)^2 (t,b) \mathrm{d}t.$$

The energy is said to dissipate along the boundary $\{x = b\}$.
(ii) (1 point) Show that for $t \geq 2(b-a)$, we have $u(t,x) = 0$ for any $x \in [a,b]$. That is, so much energy dissipated that there is nothing left.
Hint: Find the characteristic curve of function $\partial_t u + \partial_x u$.

Exercise 3. The goal of this question is to show that, in $\mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^2} \mathrm{d}x \leq 4 \int_{\mathbb{R}^3} |\nabla u(x)|^2 \mathrm{d}x, \quad \text{for all } u \in C_c^\infty(\mathbb{R}^3). \tag{1}$$

We mention here that inequality (1) is the so-called Hardy inequality.
(i) (1 point) Let $\omega \in \partial B_1(0)$. Show that

$$\int_0^\infty |u(r\omega)|^2 \mathrm{d}r = -2 \int_0^\infty u(r\omega) \left[(\omega \cdot \nabla_x u)(r\omega)\right] r \mathrm{d}r.$$

Hint: Integration by parts in $(0,\infty)$.
(ii) (1 point) Using the Cauchy-Schwarz inequality and the polar coordinates $(r,\omega) \in (0,\infty) \times \partial B_1(0)$ in $\mathbb{R}^3$ ($\mathrm{d}x = r^2 \mathrm{d}r \mathrm{d}S_\omega$) deduce the inequality (1).

Exercise 4. Let $(f,g) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3)$ and $\mathbb{R}^+ = (0,\infty)$. Recall that, the unique solution of the following Cauchy problem

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ (u, \partial_t u)_{|t=0} = (f,g), & \text{for } x \in \mathbb{R}^3, \end{cases}$$

can be written as

$$u(t,x)=\frac{1}{4\pi t^2}\int_{\partial B_t(x)}\left(f(y)+\nabla f(y)\cdot(y-x)+tg(y)\right)\mathrm{d}S_y$$

(i) (1 point) Show that the solution u can be rewritten as $u=u_1+u_2+u_3$ where

$$\begin{aligned}
u_1(t,x)&=\frac{1}{4\pi}\int_{\partial B_1(0)} f(x+t\omega)\mathrm{d}S_\omega,\\
u_2(t,x)&=\frac{t}{4\pi}\int_{\partial B_1(0)} g(x+t\omega)\mathrm{d}S_\omega,\\
u_3(t,x)&=\frac{t}{4\pi}\int_{\partial B_1(0)} \nabla f(x+t\omega)\cdot\omega\mathrm{d}S_\omega.
\end{aligned}$$

Hint: Using the change of variable $y=x+t\omega$.

(ii) (1 point) Using the Cauchy–Schwarz inequality and the polar coordinates in $\mathbb{R}^3$ prove that there exists $(C_1,C_2,C_3)\in(0,\infty)^3$ (independent of f and g) such that

$$\begin{aligned}
\int_0^\infty |u_1(t,x)|^2\mathrm{d}t&\le C_1\int_{\mathbb{R}^3}\frac{|f(x+y)|^2}{|y|^2}\mathrm{d}y \quad \text{for all } x\in\mathbb{R}^3,\\
\int_0^\infty |u_2(t,x)|^2\mathrm{d}t&\le C_2\int_{\mathbb{R}^3}|g(x+y)|^2\mathrm{d}y \quad \text{for all } x\in\mathbb{R}^3,\\
\int_0^\infty |u_3(t,x)|^2\mathrm{d}t&\le C_3\int_{\mathbb{R}^3}|\nabla f(x+y)|^2\mathrm{d}y \quad \text{for all } x\in\mathbb{R}^3.
\end{aligned}$$

Hint: Using the polar coordinates $y=t\omega$ and $\mathrm{d}y=t^2\mathrm{d}t\mathrm{d}S_\omega$.

(iii) (1 point) Deduce that there exists $C>0$ (independent of f and g) such that

$$\sup_{x\in\mathbb{R}^3}\int_0^\infty |u(t,x)|^2\mathrm{d}t\le C\int_{\mathbb{R}^3}\left(|\nabla f(y)|^2+|g(y)|^2\right)\mathrm{d}y.$$

The above inequality is a type of Morawetz inequality.

Hint: Using the Hardy inequality.